

# Pfaffian bundles on cubic surfaces and configurations of planes

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## Abstract

We give a canonical birational map between the moduli space of pfaffian vector bundles on a cubic surface and the space of complete pentahedron inscribed in the cubic surface. As a by-product, we give an explicit normal form for five general lines in  $\mathbb{P}_5$ . Applications to the geometry of the Palatini scrolls and the Debarre-Voisin hyperkähler manifolds are also discussed.

## 1 Introduction

In the classical theory of determinantal hypersurfaces, the case of pfaffian cubics has already found many applications. ([Ma-Ti], [I-R], [Dr]). This paper presents new invariants for these objects.

Let  $\mathbb{P}_5$  be a five dimensional projective space over the complex numbers, and denote by  $V_6$  the vector space  $H^0(\mathcal{O}_{\mathbb{P}_5}(1))$ . For  $n \geq 2$ , let  $\pi_{n-1}$  be a projective space of dimension  $n - 1$ , and  $W_n = H^0(\mathcal{O}_{\pi_{n-1}}(1))$ .

**Definition 1.1** *For  $2 \leq n$ , a general element of  $\bigwedge^2 V_6 \otimes W_n$  gives an exact sequence:*

$$0 \longrightarrow V_6^\vee \otimes \mathcal{O}_{\pi_{n-1}}(-1) \xrightarrow{M} V_6 \otimes \mathcal{O}_{\pi_{n-1}} \longrightarrow F \longrightarrow 0$$

*with  $M = -{}^tM$ , and where  $F$  is a rank 2 sheaf over the divisor defined by the pfaffian of  $M$ . For  $n \leq 6$ , the sheaf  $F$  will be a vector bundle over a smooth cubic divisor, and will be called a pfaffian bundle.*

It is known from the classical works on the representation of a cubic form by a pfaffian (Cf [Be]), that for  $n \geq 6$  a general cubic divisor is not a pfaffian, and for  $3 \leq n \leq 5$  the pfaffian bundles have moduli spaces of strictly positive dimension.

The section 2 studies the easy case  $n = 3$  to introduce some invariants of the pfaffian bundles. The universal situation is also described in this section. Many geometric interpretations of the next sections are specializations of this construction.

The section 3 details the case  $n = 4$ . The link with Palatini threefolds is tautological. Those varieties are well-known to be the only known examples of smooth 3 dimensional varieties  $X$  in  $\mathbb{P}_5$  such that  $h^0(\mathcal{O}_X(2)) > h^0(\mathcal{O}_{\mathbb{P}_5}(2))$ . One can find references for their Hilbert scheme ([Fa-Fa], [Fa-Me]), and also references where they are in a list of exceptions to some geometrical property (cf [Me-Po], [Ot]). But some studies for their own properties are still missing. For instance, the full description of their anticanonical system given in this section seems new. The analogy with a Veronese surface embedded in  $\mathbb{P}_4$  is often done, but in the Palatini's case, no description in terms of endomorphisms of rank one is known. The remark 3.10 gives some results in this direction.

The Grassmannian  $G(2, V_6)$  is also called the third Severi's variety and it is a classical result that its projection from a general line has a single triple point (cf [I-M], [Z]). In section 4, we show that it has a second exceptional property of this type: Its projection from a general  $\mathbb{P}_3$  has a single multiple point of order 5. From this result and the descriptions obtained in the section 3, we obtain an explicit normal form for five general lines of  $\mathbb{P}_5$  and also the following theorem:

**Definition 1.2** *A complete pentaedron inscribed in a cubic surface  $S$  of the projective space  $\pi_3$  is a set of 5 planes  $H_0, \dots, H_4$  of  $\pi_3$  such that:*

- i)  $(H_0, \dots, H_4)$  is a projective basis of  $\pi_3^\vee$ .*
- ii) The 10 points  $(H_i \cap H_j \cap H_k)_{0 \leq i < j < k \leq 4}$  are on  $S$ .*

*We define  $\mathcal{H}$  (resp.  $\mathcal{H}_{ord}$ ) to be the space of pairs  $(S, \Pi)$  such that  $S$  is a cubic surface of  $\pi_3$  and  $\Pi$  is a complete pentaedron inscribed in  $S$  (resp. complete pentahedron inscribed in  $S$  with an order on the five planes).*

**Theorem 1.3** *The space  $\mathbb{P}(W_4 \otimes \bigwedge^2 V_6) / PGL(V_6)$  is canonically birational to  $\mathcal{H}$ .*

Moreover, this map is explicited in two different ways and we have obtained a generic formula. Recently, F. Tanturri (cf [T]) found an algorithm to obtain a pfaffian representation from the equation of a cubic surface. Although some representations are similar, the main difference is that any pfaffian bundle on the surface would solve his problem, while in our situation we have additional requirements such that only one bundle is solution. (cf. the section 4.2).

In the last section we investigate those properties in families. We explain how the Debarre-Voisin's symplectic manifold can be considered as a parameter space for Palatini threefolds in a six dimensional variety of  $\mathbb{P}_9$ . Those varieties of dimension six were discovered by C. Peskine. They are of independent interest because they are smooth and non quadratically normal in  $\mathbb{P}_9$  (it's a boundary case in Zak's theory of quadratic normality). However, most of their geometric properties are unknown. In particular, it would be very interesting to understand those varieties from a Palatini threefold in a similar way that a Veronese surface is related to  $\mathbb{P}_2 \times \mathbb{P}_2$ . So we will also explain in this section the consequences on the Peskine's varieties of the work on the Palatini threefolds done in the section 3.

## 2 Invariants of Pfaffian bundles over plane cubics.

### 2.1 Ruled surfaces in $\mathbb{P}_5$ , and the $n = 3$ case.

In this section, we detail the case  $n = 3$ . The following easy lemma is a basic step that we will en-light the next sections.

**Lemma 2.1** *For a general element of  $W_3 \otimes \bigwedge^2 V_6$ , we consider the associated exact sequence:*

$$0 \longrightarrow V_6^\vee \otimes \mathcal{O}_{\pi_2}(-1) \xrightarrow{M} V_6 \otimes \mathcal{O}_{\pi_2} \longrightarrow F \longrightarrow 0 \quad (1)$$

*with  $M = -{}^tM$ . The cokernel  $F$  is a rank 2 vector bundle over the smooth plane cubic  $C$  defined by the pfaffian of  $M$ , and  $F$  is isomorphic to one of the following bundles:*

- a)  $\mathcal{L}(1) \oplus \mathcal{L}^\vee(1)$ , where  $\mathcal{L}$  is a line bundle of degree 0 on  $C$  such that  $h^0(\mathcal{L}^2) = 0$ .
- b)  $F$  is the unique unsplit extension:

$$0 \rightarrow \theta(1) \rightarrow F \rightarrow \theta(1) \rightarrow 0$$

*where  $\theta^2 = \mathcal{O}_C$  and  $\theta \neq \mathcal{O}_C$ .*

- c)  $F = \theta(1) \oplus \theta(1)$  where  $\theta^2 = \mathcal{O}_C$  and  $\theta \neq \mathcal{O}_C$ .

*Proof:* To simplify the notations, let  $F_0$  denote  $F(-1)$ . First one can remark that  $h^0(F_0) = 0$ , and that  $F_0 \simeq (F_0)^\vee$  because  $M$  is skew-symmetric. So we have  $\wedge^2(F_0) = \mathcal{O}_C$ . We choose a point  $p$  on  $C$ . We will now prove that there is a point  $r$  of  $C$  such that  $h^0(F_0(p-r)) > 0$ .

From Riemann-Roch's theorem the bundle  $F_0(p)$  has a pencil of sections. This gives, on  $\mathbb{P}_1 \times C$ , a section of the bundle  $\mathcal{O}_{\mathbb{P}_1}(1) \boxtimes F_0(p)$ . But the computation of the second Chern's class of this bundle implies that this section has a non empty vanishing locus, so there is a point  $r$  of  $C$  such that  $h^0(F_0(p-r)) > 0$ . Let's recall that  $h^0(F_0) = 0$  to obtain that  $\mathcal{O}_C(p-r)$  is not trivial and that  $F$  is isomorphic to one of the 3 above cases.  $\square$

**Remark 2.2** *The ruled surface  $\text{Proj}(S^\bullet(F))$  has a natural embedding in  $\mathbb{P}_5$  given by the surjection in the sequence (1) such that in the cases:*

- a) *it contains 2 plane cubics, and the 2 planes spanned by these curves are disjoint in  $\mathbb{P}_5$ .*
- b) *it contains only 1 plane cubic.*
- c) *it contains infinitely many plane cubics. The planes spanned by these curves are the planes of a Segre:  $\mathbb{P}_1 \times \mathbb{P}_2 \subset \mathbb{P}_5$ .*

*Moreover, the planes in those 3 cases are the planes of  $\mathbb{P}_5$  isotropic for all the skew-symmetric forms defined by  $M$ .*

*Proof:* In those 3 cases, the bundle  $F$  has an invertible quotient of rank 1 and degree 3. We just have to show that those embeddings of  $C$  are isotropic for  $M$ . But it is a corollary of the fact that the resolution of  $F$  can have a skew-symmetric form deduced from the isomorphism:  $\wedge^2(F(-1)) \simeq \mathcal{O}_C$ . Conversely, any isotropic plane for  $M$  gives the existence of  $P \in GL(V_6)$  such that:  ${}^tP.M.P = \begin{pmatrix} 0 & -{}^tA \\ A & B \end{pmatrix}$ , where  $A, B$  are 3 by 3 matrices with linear entries. So the cokernel of  $A$  gives the expected invertible quotient of  $F$  of degree 3.  $\square$

## 2.2 Universal settings and the $SL(V_6)$ -invariant double cover

**Definition 2.3** Let  $G(3, V_6^\vee)$  and  $G(3, \bigwedge^2 V_6)$  be the Grassmannians of 3-dimensional vector subspaces of  $V_6$  and  $\bigwedge^2 V_6$ . Denote by  $K_3$  and  $R_3$  their tautological sub-bundles. We define the isotropic incidence:

$$\begin{array}{ccc} Z \subset G(3, V_6^\vee) \times G(3, \bigwedge^2 V_6) & \xrightarrow{p_2} & G(3, \bigwedge^2 V_6) \\ p_1 \downarrow & & \\ & & G(3, V_6^\vee) \end{array}$$

to be the vanishing locus of the unique  $SL(V_6)$ -invariant section of  $\bigwedge^2 K_3^\vee \boxtimes R_3^\vee$ . Denote by  $\mathcal{U}$  the open subset of  $G(3, \bigwedge^2 V_6)$  made of subspaces such that the intersection of their projectivisation with the pfaffian hypersurface of  $\mathbb{P}(\bigwedge^2 V_6)$  is a smooth cubic curve.

The restriction of  $Z$  to  $G(3, V_6^\vee) \times \mathcal{U}$  will be noted:  $Z_{\mathcal{U}}$ . Let  $E_{12}$  be the rank 12 bundle defined by the exact sequence:

$$0 \longrightarrow E_{12} \longrightarrow \bigwedge^2 V_6 \otimes \mathcal{O}_{G(3, V_6^\vee)} \longrightarrow \bigwedge^2 K_3^\vee \longrightarrow 0 \quad (2)$$

I'd like to thanks A. Kuznetsov for the following description of  $Z$  from the relative Grassmannian.

**Proposition 2.4** The isotropic incidence  $Z$  is isomorphic to the relative Grassmannian  $G(3, E_{12})$  of linear subspaces of the bundle  $E_{12}$ . The projection  $Z_{\mathcal{U}} \rightarrow \mathcal{U} \subset G(3, \bigwedge^2 V_6)$  is generically finite of degree 2. The fibers of this morphism over an element of type  $a, b, c$  in Lemma 2.1 is respectively in  $G(3, V_6^\vee)$ : 2 points, 1 point, and a rational cubic curve.

*Proof:* Let  $(\mu, \nu)$  be an element of  $G(3, V_6^\vee) \times G(3, \bigwedge^2 V_6)$ . The fiber of a vector bundle at  $\mu$  (resp.  $\nu$ ) will be noted by the name of the bundle with the index  $\mu$  (resp.  $\nu$ ). The vector space  $K_{3, \mu}$  is isotropic for all the skew-symmetric forms defined by the elements of  $R_{3, \nu}$  if and only if  $(\mu, \nu) \in Z$ , but also if and only if the composition:

$$R_{3, \nu} \longrightarrow \bigwedge^2 V_6 \longrightarrow \bigwedge^2 K_{3, \mu}^\vee$$

is the zero map. So  $(\mu, \nu) \in Z \iff R_{3,\nu} \subset E_{12,\mu}$  and we have the equality  $Z = G(3, E_{12})$ .

The end of the assertion follows immediatly from the Lemma 2.1 and the Remark 2.2.  $\square$

**Corollary 2.5** *The locus  $\mathcal{U}_c$  in  $\mathcal{U} \subset G(3, \bigwedge^2 V_6)$  of planes of type c) has codimension 3. Consider the following relation on  $\mathcal{U}_c$ :  $p\mathcal{R}p'$  if and only if  $p_1(p_2^{-1}(p)) = p_1(p_2^{-1}(p'))$ .*

*For any element  $p$  of  $\mathcal{U}_c$ , there is a six dimensional subspace  $L_p$  of  $\bigwedge^2 V_6$  such that the equivalence class of  $p$  for  $\mathcal{R}$  is an openset of  $G(3, L_p)$ .*

*Proof:* From the proposition 2.4, for any  $p$  in  $\mathcal{U}_c$ ,  $p_1(p_2^{-1}(p))$  is a smooth rational cubic curve  $C_p$  in  $G(3, V_6^\vee)$ . So the restriction of  $E_{12}$  to  $C_p$  is  $6\mathcal{O}_{\mathbb{P}_1} \oplus 6\mathcal{O}_{\mathbb{P}_1}(-1)$ , and this bundle has a natural trivial sub-bundle of rank 6. Let  $L_p$  be the six dimensional vector subspace of  $\bigwedge^2 V_6$  obtained from the image of this sub-bundle by the injection of the sequence (2).

The proposition 2.4 describes  $p_1^{-1}(C_p)$  as the relative Grassmannian  $G(3, E_{12|C_p})$ . Let  $F$  be a subvector bundle of rank 3 of  $E_{12|C_p} = L_p \otimes \mathcal{O}_{\mathbb{P}_1} \oplus 6\mathcal{O}_{\mathbb{P}_1}(-1)$ . The case c) appears when the line bundle  $\wedge^3 F^\vee$  contracts the curve  $C_p$ . But  $\wedge^3 F^\vee$  is not ample if and only if  $F$  is a trivial sub-bundle of  $L_p \otimes \mathcal{O}_{\mathbb{P}_1}$ . So  $p_1^{-1}(C_p) \cap p_2^{-1}(\mathcal{U}_c)$  is  $(\mathcal{U} \cap G(3, L_p)) \times C_p$ , and the equivalence classe of  $p$  for  $\mathcal{R}$  is  $\mathcal{U} \cap G(3, L_p)$ . So the dimension of  $\mathcal{U}_c$  is the sum of the dimension of  $G(3, 6)$  with the dimension of the family of rational cubic curves in  $G(3, V_6^\vee)$ . In conclusion  $\mathcal{U}_c$  has dimension 33 and codimension 3 in  $G(3, \bigwedge^2 V_6)$ .  $\square$

### 3 Palatini threefolds

In this section we will study the case  $n = 4$ .

#### 3.1 Définition and classical properties

**Definition 3.1** *A smooth 3 dimensional sub-variety  $X$  of  $\mathbb{P}_5$  is called a Palatini threefold<sup>1</sup> if there exists an element of  $\alpha \in \bigwedge^2 V_6 \otimes W_4$  such that  $X = \text{Proj}(S^\bullet(F))$  where  $F$  is the pfaffian vector bundle defined from  $\alpha$  in the Definition 1.1 with  $n = 4$ .*

**Notation 3.2** *In this section, denote by  $X$  a Palatini threefold in  $\mathbb{P}_5$ , by  $h$  the class of an hyperplane of  $\mathbb{P}_5$ , by  $S$  the pfaffian cubic surface in  $\pi_3$  and by  $s$  the pullback on  $X$  of the class of an hyperplane of  $\pi_3$ . The cotangent bundle of  $\mathbb{P}_5$  will be noted  $\Omega_{\mathbb{P}_5}^1$ .*

So we can immediately obtain the well known resolution of its ideal:

**Remark 3.3** *The ideal  $I_X$  of a Palatini threefold  $X$  in  $\mathbb{P}_5$  has the following resolution:*

$$0 \longrightarrow W_4^\vee \otimes \mathcal{O}_{\mathbb{P}_5} \xrightarrow{\alpha} \Omega_{\mathbb{P}_5}^1(2h) \longrightarrow I_X(4h) \longrightarrow 0 \quad (3)$$

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<sup>1</sup>or a Palatini scroll

and the famous equality:

$$h^0 \mathcal{O}_X(2h) = h^0 \mathcal{O}_{\mathbb{P}_5}(2h) + 1.$$

To explain the natural embedding of  $X$  in the point/plane incidence of  $\mathbb{P}_5$ , F. Zak introduced the following vector bundle:

**Definition 3.4** *The canonical extension*

$$0 \longrightarrow \Omega_{\mathbb{P}_5}^1(h) \longrightarrow V_6 \otimes \mathcal{O}_{\mathbb{P}_5} \longrightarrow \mathcal{O}_{\mathbb{P}_5}(h) \longrightarrow 0$$

induces on a Palatini threefold  $X$  the following one, where the rank 3 vector bundle  $E_X$  is defined to be the middle term.

$$0 \longrightarrow N_X^\vee(3h) \longrightarrow E_X \longrightarrow \mathcal{O}_X(h) \longrightarrow 0.$$

The resolution of  $E_X$  as an  $\mathcal{O}_{\mathbb{P}_5}$ -module is:

$$0 \longrightarrow W_4^\vee \otimes \mathcal{O}_{\mathbb{P}_5}(-h) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_{\mathbb{P}_5} \longrightarrow E_X \longrightarrow 0. \quad (4)$$

and the determinant of  $E_X$  is  $\mathcal{O}_X(3h - s)$ .

From the inclusion  $W_4^\vee \subset \bigwedge^2 V_6$  and the identification  $W_4 = \wedge^3 W_4^\vee$ , we can consider  $\pi_3^\vee$  as a sub-variety of  $G(3, \bigwedge^2 V_6)$ .

**Proposition 3.5** *Let  $Z_4$  be the restriction of the isotropic incidence  $Z \subset G(3, V_6^\vee) \times G(3, \bigwedge^2 V_6)$  to  $G(3, V_6^\vee) \times \pi_3^\vee$ . Then  $Z_4$  is isomorphic to  $X$  and the projection from  $Z_4$  to  $G(3, V_6^\vee)$  is an embedding.*

*Proof:* Let's first recall the classical description of quadrisecant lines to  $X$ . Let  $A^\vee$  and  $B$  be the 3 dimensional vector subspaces of  $V_6^\vee$  and  $W_4^\vee$  corresponding to a point of  $Z_4$ . Denote by  $A'$  the kernel of the sujection from  $V_6$  to  $A$  and  $\mathbb{P}(A^\vee) \subset \mathbb{P}_5$  by  $\pi_A$ . The restriction of  $\Omega_{\mathbb{P}_5}^1(1)$  to  $\pi_A$  is  $A' \otimes \mathcal{O}_{\pi_A} \oplus \Omega_{\pi_A}^1(1)$ . Now using the isotropy of  $\pi_A$  with respect to all the elements of  $B$  we see that the restriction to  $\pi_A$  of the left map of sequence (3) is the direct sum of the following maps:

$$B \otimes \mathcal{O}_{\pi_A}(-1) \rightarrow A' \otimes \mathcal{O}_{\pi_A} \text{ and } \frac{W_4^\vee}{B} \otimes \mathcal{O}_{\pi_A}(-1) \rightarrow \Omega_{\pi_A}^1(1).$$

The determinant of the first one gives a cubic curve in  $\pi_A \cap X$ , and the second one vanishes on a single (residual) point  $\mu$  of  $\pi_A \cap X$ . So we have constructed a morphism from  $Z_4$  to  $X$ :  $(A^\vee, B) \mapsto \mu$ .

Moreover, this vanishing shows by specialization of the sequence (4) at the point  $\mu$  that the fiber of  $E_X^\vee$  at  $\mu$  is  $A^\vee$ . So  $Z_4$  and  $X$  have the same image in  $G(3, V_6^\vee)$ , and the proof of the proposition is reduced to the proof of the embedding of  $Z_4$  to  $G(3, V_6^\vee)$ . But the fiber of this morphism over the point of  $G(3, V_6^\vee)$  corresponding to  $A^\vee$  is a single point because  $A^\vee$  is not isotropic for all the elements of  $W_4^\vee$ . So this projection of  $Z_4$  is one to one, and it must be an isomorphism because the fibers are given by linear conditions.  $\square$

### 3.2 Anticanonical properties

**Lemma 3.6** *The canonical line bundle of  $X$ :  $\omega_X$  is isomorphic to  $\mathcal{O}_X(s - 2h)$ . With the notations 3.2, we have from the equality  $W_4 = H^0(\mathcal{O}_S(1))$  a canonical isomorphism:*

$$H^0(\omega_X^\vee) = W_4^\vee$$

*Proof:* The isomorphism  $\omega_X \simeq \mathcal{O}_X(s - 2h)$  can be computed directly from the definition 3.1. We obtain the isomorphism  $H^0(\omega_X^\vee) = W_4^\vee$  from the isomorphism between  $X$  and  $Z_4$  found in the proposition 3.5 and the fact that  $\omega_{Z_4}^\vee$  is the pull back of  $\mathcal{O}_{\pi_3^\vee}(1)$ .  $\square$

**Proposition 3.7** *The linear system  $|\omega_X^\vee|$  has no base points and gives a morphism of degree 2:*

$$X \xrightarrow{2:1} \pi_3^\vee \subset G(3, \bigwedge^2 V_6)$$

*The anticanonical linear system of  $X$  contracts 5 rational curves of degree 3 for the embeddings of  $X$  in  $\mathbb{P}_5$  and in  $G(3, V_6^\vee)$ .*

*Proof:* The contracted curves of this morphism correspond to the case c) of the lemma 2.2 namely the planes of a segre. So they are smooth rational cubic curves in  $G(3, V_6^\vee)$ . By definition, on such a curve, the divisors  $2h$  and  $s$  are equivalent because  $\omega_X^\vee = \mathcal{O}_X(2h - s)$ . So those curves have the same degree with respect to  $h$  than to  $3h - s$ . So the proof will end after the following:

**Lemma 3.8** *Let  $\bar{F}$  be the normalized bundle  $F(-1)$ . The vector space*

$$H^\vee = H^1((S^2 \bar{F})(-1))$$

*has dimension 5 and it is the kernel of the 4 by 4 pfaffian map associated to  $M$ :*

$$0 \longrightarrow H^\vee \longrightarrow \bigwedge^2 V_6 = \bigwedge^4 V_6^\vee \longrightarrow S^2 W_4 \longrightarrow 0.$$

*Moreover the exceptional locus in  $\pi_3^\vee$  of the projection  $X = Z_4 \rightarrow \pi_3^\vee$  is given by the 4 by 4 pfaffians of a skew-symmetric map:*

$$H^\vee \otimes \mathcal{O}_{\pi_3^\vee}(-1) \longrightarrow H \otimes \mathcal{O}_{\pi_3^\vee}. \quad (5)$$

*Proof:* Let  $i$  be an isomorphism:  $\wedge^2 \bar{F} \rightarrow \mathcal{O}_S$ . The restriction of  $F$  to a plane  $P$  is of type c) in the lemma 2.2 if and only if we have  $h^1(S^2(\bar{F}_P)) = 3$ .

To globalize this condition, let's consider the complex:

$$C^\bullet : 0 \longrightarrow V_6^\vee \otimes \mathcal{O}_{\pi_3}(-2) \xrightarrow{M} V_6 \otimes \mathcal{O}_{\pi_3}(-1) \longrightarrow 0$$

exact in degree  $-1$  and with cohomology  $\bar{F}$  in degree 0. The exterior power of  $C^\bullet$  tensorised by  $\mathcal{O}_{\pi_3}(2)$  is:

$$0 \longrightarrow S^2 V_6^\vee \otimes \mathcal{O}_{\pi_3}(-2) \longrightarrow V_6^\vee \otimes V_6 \otimes \mathcal{O}_{\pi_3}(-1) \longrightarrow \bigwedge^2 V_6 \otimes \mathcal{O}_{\pi_3} \longrightarrow 0$$

with cohomology in degree  $(-2, -1, 0)$ :  $(0, S^2(\bar{F})(-1), (\wedge^2(\bar{F}))(2))$ . So the hypercohomology's spectral sequence of this complex gives the exact sequence (5) and the dimension of  $H^\vee$ .

Now take the symmetric power of  $C^\bullet$  :

$$K^\bullet : 0 \longrightarrow \bigwedge^2 V_6^\vee \otimes \mathcal{O}_{\pi_3}(-4) \longrightarrow V_6^\vee \otimes V_6 \otimes \mathcal{O}_{\pi_3}(-3) \longrightarrow S^2 V_6 \otimes \mathcal{O}_{\pi_3}(-2) \longrightarrow 0.$$

Its cohomology in degree  $(-2, -1, 0)$  is  $(0, (\wedge^2(\bar{F}))(-3), S^2(\bar{F}))$ .

Now consider the point/plane incidence variety  $I_3 \subset \pi_3^\vee \times \pi_3$  and denote by  $p_3^\vee$  and  $p_3$  the first and second projections of this product. The hyper-direct image by  $p_3^\vee$  of  $p_{3*} K^\bullet$  gives the exact sequence:

$$0 \rightarrow p_{3*}^\vee(p_3^*(S^2 \bar{F})) \rightarrow H^1(S^2(\bar{F})(-1)) \otimes \mathcal{O}_{\pi_3^\vee}(-1) \xrightarrow{d_M} H^1(S^2(\bar{F})) \otimes \mathcal{O}_{\pi_3^\vee} \rightarrow R^1 p_{3*}^\vee(p_3^*(S^2 \bar{F})) \rightarrow 0.$$

Let's now explain how to consider the map  $d_M$  as a skew-symmetric map. The isomorphism  $i$  gives a symmetric isomorphism  $i' : S_2(\bar{F}) \rightarrow S_2(\bar{F}^\vee)$  so the following square is commutative:

$$\begin{array}{ccc} (S_2 \bar{F})(-1) \otimes S_2 \bar{F} & \xrightarrow{i' \otimes id} & (S_2 \bar{F}^\vee)(-1) \otimes S_2 \bar{F} \\ id \otimes i' \downarrow & & \downarrow \tau \\ (S_2 \bar{F})(-1) \otimes S_2 \bar{F}^\vee & \xrightarrow{\tau'} & \mathcal{O}_S(-1) \end{array}.$$

The cup-product  $H^1((S_2 \bar{F})(-1)) \otimes H^1((S_2 \bar{F})(-1)) \rightarrow H^2((S_2 \bar{F} \otimes S_2 \bar{F})(-2))$  is anti-commutative, so for any  $z \in W_4$  the following square is also anti-commutative:

$$\begin{array}{ccc} H^1((S_2 \bar{F})(-1)) \otimes H^1((S_2 \bar{F})(-1)) & \xrightarrow{d_{M,z} \otimes id} & H^1(S_2 \bar{F}) \otimes H^1((S_2 \bar{F})(-1)) \\ id \otimes d_{M,z} \downarrow & & \downarrow \cup \\ H^1((S_2 \bar{F})(-1)) \otimes H^1(S_2 \bar{F}) & & H^2(S_2 \bar{F} \otimes (S_2 \bar{F})(-1)) \\ \cup \downarrow & & \downarrow \overline{\tau \circ (i' \otimes id)} \\ H^2((S_2 \bar{F})(-1) \otimes S_2 \bar{F}) & \xrightarrow{\overline{\tau' \circ (id \otimes i')}} & H^2(\mathcal{O}_S(-1)) \end{array}.$$

In conclusion, the composition:

$$H^\vee \otimes \mathcal{O}_{\pi_3^\vee}(-1) \xrightarrow{d_M} H^1(S_2 \bar{F}) \otimes \mathcal{O}_{\pi_3^\vee} \xrightarrow{\bar{i}'} H^1(S_2(\bar{F}^\vee) \otimes \mathcal{O}_{\pi_3^\vee}) \xrightarrow{\text{Serre's duality}} H \otimes \mathcal{O}_{\pi_3^\vee}$$

is skew-symmetric and the lemma is proved because the type c) cases correspond to the locus where this map has rank at most 2.  $\square$

**Definition 3.9** Let  $\Sigma_5$  be the symmetric product of order 5 of  $\pi_3^\vee$ . We define the rational map  $\Phi_1$  to be:

$$\begin{array}{ccc} \Phi_1 : \mathbb{P}(W_4 \otimes \bigwedge^2 V_6) / PGL(V_6) & \dashrightarrow & \mathbb{P}(S^3(W_4)) \times \Sigma_5 \\ \alpha & \mapsto & (S, (h_0 \dots h_4)) \end{array}$$

where  $S$  is the pfaffian cubic surface defined by  $M_\alpha$ , and where  $h_0, \dots, h_4$  are the five linear sections of  $S$  defined in the proposition 3.7.

In the section 4 we will understand the image of this map.



### 3.3 Palatini threefolds and endomorphisms

The exceptional geometric properties of a Palatini threefold are classically considered as natural generalizations of what happens to a Veronese  $\mathcal{V}$  surface embedded in  $\mathbb{P}_4$ . For instance, in the Veronese situation, the sequence 3 is replaced by:

$$0 \longrightarrow W_3^\vee \otimes \mathcal{O}_{\mathbb{P}_4} \xrightarrow{\alpha} \Omega_{\mathbb{P}_4}^1(2h) \longrightarrow I_{\mathcal{V}}(3h) \longrightarrow 0.$$

But the main difference is that in the theory of Severi varieties the embedding of  $\mathcal{V}$  by the complete linear system  $|\mathcal{O}_{\mathcal{V}}(h)|$  is understood from an interpretation in terms of matrices of size  $3 \times 3$  of rank 1. For a Palatini threefold, there is no similar result to describe the embedding by the complete linear system  $|\mathcal{O}_X(2h)|$ . The following remark could be a first step in this direction:

**Remark 3.10** *The restriction of the line bundle  $\omega_X^\vee \boxtimes \mathcal{O}_X(s)$  to the diagonal of  $X \times X$  gives the natural inclusions:*

$$W_4^\vee \otimes W_4 \subset H^0(\mathcal{O}_X(2h))$$

*In other words, the embedding of a Palatini threefold  $X$  with  $|\mathcal{O}_X(2h)|$  has a canonical projection in  $\mathbb{P}(W_4^\vee \otimes W_4)$ , and the image of  $X$  by this projection is included in the endomorphisms of  $W_4$  of rank 1.*

*Proof:* It's straightforward from the lemma 3.6.  $\square$

## 4 Geometry in $\bigwedge^2 V_6$

### 4.1 Projections from linear spaces

The Grassmannian variety  $G(2,6)$  is one of the 4 Severi varieties, and well known to have the exceptional property that its projection from a general line has a unique triple point. Here, we prove that it has the same property with projection from a  $\mathbb{P}_3$  and points of multiplicity 5:

**Proposition 4.1** *Denote by  $\mathcal{U}_5$  the subspace of  $G(5, \bigwedge^2 V_6)$  defined by the five dimensional vector spaces such that the intersection of their projectivisation with  $G(2, V_6)$  is 5 linearly independent distinct points. Let  $W_4^\vee$  be a general subspace of  $\bigwedge^2 V_6$ , then there is a unique element of  $\mathcal{U}_5$  containing  $W_4^\vee$ .*

*Proof:* First remark that the incidence variety

$$I_{4,5} = \{(W_4^\vee, W_5^\vee) | W_4^\vee \subset W_5^\vee \subset \bigwedge^2 V_6, W_5^\vee \in \mathcal{U}_5\}$$

has the same dimension as  $G(4, \bigwedge^2 V_6)$ , so we have to prove that the natural projection is birational.

So, consider a general element  $W_4^\vee$  in the image of this projection, and chose an element  $W_5^\vee$  such that  $(W_4^\vee, W_5^\vee) \in I_{4,5}$ , and denote by  $\pi_3, \pi_4$  their projectivisation as in the introduction. The vector space  $H^0(I_{\pi_3 \cup G(2, V_6)}(2))$  is the kernel of the pfaffian map  $\bigwedge^4 V_6^\vee \rightarrow S_2 W_4$ . So it has dimension 5. Now remark that for such a  $W_5^\vee$  we also have  $h^0(I_{\pi_4 \cup G(2, V_6)}(2)) = 5$  because the ideal of those 5 points in  $\pi_4$  is a 10 dimensional space of quadrics. So we proved that  $\pi_4$  must be in all the quadrics of  $H^0(I_{\pi_3 \cup G(2, V_6)}(2))$ . So we have found the following linear conditions satisfied by any  $W_5^\vee$  of  $\mathcal{U}_5$  containing  $W_4^\vee$ :

$$W_5^\vee \subset \bigcap_{q \in H^0(I_{\pi_3 \cup G(2, V_6)}(2))} (W_4^\vee)^{\perp_q}$$

where  $\perp_q$  denotes the orthogonal with respect to  $q$  considered as a quadratic form on  $\bigwedge^2 V_6$ . So the unicity of  $W_5^\vee$  will be a corollary of the existence of an exemple of  $W_4$  such that  $\bigcap_{q \in H^0(I_{\pi_3 \cup G(2, V_6)}(2))} (W_4^\vee)^{\perp_q}$  has dimension 5 as it is the case in the following:

**Example 4.2** *Let's consider a basis  $(\epsilon_i)$  of  $V_6$ , and the 5 elements*

$$u_0 = \epsilon_0 \wedge \epsilon_3, u_1 = \epsilon_1 \wedge \epsilon_4, u_2 = \epsilon_2 \wedge \epsilon_5, u_3 = (\epsilon_0 + \epsilon_1 + \epsilon_2) \wedge (\epsilon_4 + \epsilon_3 + \epsilon_5), u_4 = (\epsilon_1 + \epsilon_4 + \epsilon_2) \wedge (\epsilon_3 + \epsilon_1 + \epsilon_5).$$

*Denote by  $W_5^\vee$  the 5 dimensional vector space spanned by the  $(u_i)$  and*

$$W_4^\vee = \left\{ \sum_{0 \leq i \leq 4} \lambda_i \cdot u_i \mid \sum_{0 \leq i \leq 4} \lambda_i = 0 \right\}.$$

*Then  $\bigcap_{q \in H^0(I_{\pi_3 \cup G(2, V_6)}(2))} (W_4^\vee)^{\perp_q}$  has dimension 5.*

*Proof:* We can compute with [Macaulay2] that  $H^0(I_{\pi_3 \cup G(2, V_6)}(2))$  is generated by the five quadrics in Plucker coordinates:

$$\begin{aligned} & \cdot p_{(3,4)}p_{(1,5)} - p_{(1,4)}p_{(3,5)} + p_{(1,3)}p_{(4,5)} \\ & \cdot p_{(1,2)}p_{(0,5)} - p_{(2,4)}p_{(0,5)} - p_{(0,2)}p_{(1,5)} + p_{(2,3)}p_{(1,5)} + p_{(0,1)}p_{(2,5)} - p_{(1,3)}p_{(2,5)} + p_{(0,4)}p_{(2,5)} - \\ & \quad p_{(3,4)}p_{(2,5)} + p_{(1,2)}p_{(3,5)} + p_{(2,4)}p_{(3,5)} - p_{(0,2)}p_{(4,5)} - p_{(2,3)}p_{(4,5)} \\ & \cdot p_{(2,3)}p_{(0,4)} - p_{(0,3)}p_{(2,4)} + p_{(0,2)}p_{(3,4)} - p_{(1,3)}p_{(0,5)} + p_{(2,4)}p_{(0,5)} - p_{(3,4)}p_{(0,5)} + p_{(0,3)}p_{(1,5)} - \\ & \quad p_{(2,3)}p_{(1,5)} + p_{(1,3)}p_{(2,5)} - p_{(0,4)}p_{(2,5)} + p_{(3,4)}p_{(2,5)} - p_{(0,1)}p_{(3,5)} - p_{(1,2)}p_{(3,5)} + p_{(0,4)}p_{(3,5)} - \\ & \quad p_{(2,4)}p_{(3,5)} + p_{(0,2)}p_{(4,5)} - p_{(0,3)}p_{(4,5)} + p_{(2,3)}p_{(4,5)} \\ & \cdot p_{(1,2)}p_{(0,4)} - p_{(0,2)}p_{(1,4)} + p_{(0,1)}p_{(2,4)} - p_{(2,4)}p_{(0,5)} + p_{(2,3)}p_{(1,5)} - p_{(1,3)}p_{(2,5)} + p_{(0,4)}p_{(2,5)} - \\ & \quad p_{(3,4)}p_{(2,5)} + p_{(1,2)}p_{(3,5)} + p_{(2,4)}p_{(3,5)} - p_{(0,2)}p_{(4,5)} - p_{(2,3)}p_{(4,5)} \\ & \cdot p_{(1,2)}p_{(0,3)} - p_{(0,2)}p_{(1,3)} + p_{(0,1)}p_{(2,3)} - p_{(2,4)}p_{(0,5)} + p_{(3,4)}p_{(0,5)} + p_{(2,3)}p_{(1,5)} - p_{(1,3)}p_{(2,5)} + \\ & \quad p_{(0,4)}p_{(2,5)} - p_{(3,4)}p_{(2,5)} + p_{(1,2)}p_{(3,5)} - p_{(0,4)}p_{(3,5)} + p_{(2,4)}p_{(3,5)} - p_{(0,2)}p_{(4,5)} + p_{(0,3)}p_{(4,5)} - \\ & \quad p_{(2,3)}p_{(4,5)} \end{aligned}$$

and check that the ideal of the orthogonal of  $\pi_3$  with respect to these 5 quadrics is generated by the 10 independant equations:  $(p_{(3,5)}, p_{(0,5)} - p_{(1,5)} + p_{(4,5)}, p_{(3,4)} + p_{(4,5)}, p_{(2,4)} - p_{(1,5)} + p_{(4,5)}, p_{(0,4)} - p_{(1,5)} + p_{(4,5)}, p_{(2,3)} - p_{(1,5)}, p_{(1,3)} - p_{(1,5)}, p_{(1,2)} + p_{(4,5)}, p_{(0,2)}, p_{(0,1)})$ . So this example completes the proof of the birationality of the projection from  $I_{4,5}$  to  $G(4, \bigwedge^2 V_6)$ , so of the proposition 4.1.  $\square$

**Corollary 4.3** *With the notations of the definition 1.2, we can define the rational map  $\Phi_2$  to be:*

$$\begin{array}{ccc} \Phi_2 : \mathbb{P}(W_4 \otimes \bigwedge^2 V_6) / PGL(V_6) & \dashrightarrow & \mathcal{H} \\ \alpha & \mapsto & (S, (H_0 \dots H_4)) \end{array}$$

where  $S$  is the pfaffian cubic surface defined by  $M_\alpha$ ,  $H_i$  is the intersection of  $\pi_3$  with the projective spaces spanned by  $(u_j)_{0 \leq j \leq 4, j \neq i}$ , and  $(u_i)_{0 \leq i \leq 4}$  are the five points of  $G(2, V_6)$  such that  $\pi_3$  is in their linear span.

*Proof:* After the proposition 4.1, we only have to explain why  $H_0, \dots, H_4$  is inscribed on  $S$ . But for  $\{i_0, \dots, i_4\} = \{0, \dots, 4\}$  the point  $H_{i_0} \cap H_{i_1} \cap H_{i_2}$  is on the line  $(u_{i_3}, u_{i_4})$  so it corresponds to a matrix of rank 4 and is on  $S$ .  $\square$

## 4.2 An explicit formula and the proof of the theorem 1.3

Surprisingly, we are able to give in this section an explicit formula. Recently, a explicit result was also found by F. Tanturri in [T]: An algorithm to obtain a pfaffian representation from a cubic equation. The two main difference, are the following:

-first he wants to find any pfaffian representation of  $S$ , but here we need to find a unique point in the moduli space.

-The construction starts with five points on  $S$ , so it is a problem of extending the 5 by 5 skew-symmetric matrix of the resolution of the 5 points to a 6 by 6 one with pfaffian  $S$ , while we start with an inscribed pentahedron.

**Lemma 4.4** *Let  $(x_i)_{0 \leq i \leq 3}$  be a basis of  $W_4$ , and  $\mathcal{A}_9$  be the following subspace of  $\mathbb{C}^{10} \times \mathbb{P}_4$ :*

$$\mathcal{A}_9 = \left\{ ((a_{i,j,k})_{0 \leq i < j < k \leq 4}, (b_i)_{0 \leq i \leq 4}) \left| \begin{array}{l} a_{0,1,4} = 1 \text{ and for } 0 \leq i \leq 4, b_i \neq 0, \\ \text{and for } 0 \leq i < j < k \leq 3, a_{i,j,k} = 1 \end{array} \right. \right\}.$$

*Then the following map is birational:*

$$\begin{array}{ccc} PGL_4 \times \mathcal{A}_9 & \rightarrow & \mathcal{H}_{ord} \\ (P, ((a_{i,j,k})_{0 \leq i < j < k \leq 4}, (b_i)_{0 \leq i \leq 4})) & \mapsto & (S, (H_0, \dots, H_4)) \end{array} \quad (6)$$

where

$$\sum_{0 \leq i < j < k \leq 4} a_{i,j,k} \cdot w_i \cdot w_j \cdot w_k = 0,$$

is an equation of  $S$ , and for all  $0 \leq i \leq 4$ ,  $w_i = 0$  is an equation of  $H_i$  with the following

$$\text{equalities: } w_4 = \sum_{i=0}^3 \frac{b_4 \cdot w_i}{b_i}, \quad \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = P \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

*Proof:* Let  $\Pi = (H_0, \dots, H_4)$  be an ordered pentahedron and  $P'$  be the unique projective transformation that sends the ordered pentahedron  $(x_0, x_1, x_2, x_3, x_0 + x_1 + x_2 + x_3)$  to  $(H_0, \dots, H_4)$ . Denote by  $h_i$  the equation of  $H_i$  defined by:  $\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = P' \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$  and  $h_4 = \sum_{i=0}^3 h_i$ . The cubic surfaces  $S$  such that  $(S, \Pi)$  is in  $\mathcal{H}_{ord}$  are the smooth surfaces with equation:

$$\sum_{0 \leq i < j < k \leq 4} A_{i,j,k} \cdot h_i \cdot h_j \cdot h_k = 0, \quad (A_{i,j,k})_{(0 \leq i < j < k \leq 4)} \in \mathbb{P}_9.$$

Now remark that the map:

$$\begin{aligned} \mathcal{A}_9 &\rightarrow \mathbb{P}_9 \\ (a, b) &\mapsto (A_{i,j,k} = a_{i,j,k} \cdot b_i \cdot b_j \cdot b_k)_{0 \leq i < j < k \leq 4} \end{aligned}$$

is birational because we can compute its inverse with the formulas<sup>2</sup>:

$$\frac{b_0}{b_3} = \frac{A_{0,1,2}}{A_{1,2,3}}, \frac{b_1}{b_3} = \frac{A_{0,1,2}}{A_{0,2,3}}, \frac{b_2}{b_3} = \frac{A_{0,1,2}}{A_{0,1,3}}, \frac{b_4}{b_3} = \frac{A_{0,1,4}}{A_{0,1,3}}, a_{i,j,4} = \frac{A_{i,j,4}}{A_{i,j,3}} \cdot \frac{A_{0,1,3}}{A_{0,1,4}}.$$

So we obtain the lemma with the equalities:  $0 \leq i \leq 4, w_i = h_i \cdot b_i$  and  $P$  defined by the product of the diagonal matrix  $(\frac{b_0}{b_4}, \dots, \frac{b_3}{b_4})$  with  $P'$ .  $\square$

**Definition 4.5** Let  $\mathcal{A}'_9$  be the set of triples  $(a, b, u)$  such that  $(a, b)$  is in  $\mathcal{A}_9$  and  $u$  is a root of the equation in  $X$ :

$$X^2 + X \cdot (1 + a_{0,2,4} - a_{0,3,4}) + a_{0,2,4} = 0$$

and denote by  $v = -(1 + a_{0,2,4} - a_{0,3,4}) - u$  the other one. Define the following:

$$e_1 = a_{0,2,4} + a_{1,2,4} - a_{2,3,4}, e_2 = 1 + a_{1,2,4} - a_{1,3,4}, e_3 = (-a_{1,2,4} + a_{1,3,4} - 1)v - a_{1,2,4} - a_{0,2,4} + a_{2,3,4}$$

$$M_4 = \begin{pmatrix} 0 & u & -1 & a_{1,2,4} & e_1 & e_2 \\ -u & 0 & 0 & 0 & a_{0,2,4} & -u \\ 1 & 0 & 0 & 0 & -v & 1 \\ -a_{1,2,4} & 0 & 0 & 0 & a_{1,2,4}v & -a_{1,2,4} \\ -e_1 & -a_{0,2,4} & v & -a_{1,2,4}v & 0 & e_3 \\ -e_2 & u & -1 & a_{1,2,4} & -e_3 & 0 \end{pmatrix}$$

$$M_{0123} = \begin{pmatrix} 0 & 0 & 0 & w_0 + w_3 & w_3 & w_3 \\ 0 & 0 & 0 & w_3 & w_1 + w_3 & w_3 \\ 0 & 0 & 0 & w_3 & w_3 & w_2 + w_3 \\ -w_0 - w_3 & -w_3 & -w_3 & 0 & 0 & 0 \\ -w_3 & -w_1 - w_3 & -w_3 & 0 & 0 & 0 \\ -w_3 & -w_3 & -w_2 - w_3 & 0 & 0 & 0 \end{pmatrix}$$

---

<sup>2</sup>If one works with affine spaces instead of  $\mathbb{P}_9$  and  $\mathbb{P}_4$ , then one needs to extract a cubic root to solve the equalities.

**Theorem 4.6** For a generic element  $(P, (a, b, u))$  of  $PGL_4 \times \mathcal{A}'_9$ , the element  $\alpha \in \mathbb{P}(W_4 \otimes \bigwedge^2 V_6)$  defined by  $M = M_{0123} + w_4 M_4$  is such that:  $\Phi_1(\alpha) = \Phi_2(\alpha) = (S, \Pi)$  where the equation of  $S$  and  $\Pi$  are given by the formula in the lemma 4.4.

*Proof:* The difficulty was to find  $M_4$ . It was done by tracking the rational cubic curve in  $\mathbb{P}_5$  associated to the plane  $w_4 = 0$  in the proposition 3.7. But now that we have found  $M_4$ , it is much easier to check that  $M$  satisfies the required properties.

NB: To obtain a more compact presentation, we have glued the indexes of the  $a_{i,j,k}$  in the next formulas.

- First, one can check that the pfaffian of  $M$  is

$$a_{024}w_0w_2w_4 + a_{034}w_0w_3w_4 + a_{234}w_2w_3w_4 + a_{124}w_1w_2w_4 + a_{134}w_1w_3w_4 + w_0w_1w_4 + \sum_{0 \leq i < j < k \leq 3} w_iw_jw_k$$

- Now to prove that  $\Phi_2(\alpha) = (S, \Pi)$  we just have to remark that  $M_4$  has rank 2, and also the 4 values of  $M_{0123}$  at the points where  $(w_0, w_1, w_2, w_3)$  take the values  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ ,  $(0, 0, 0, 1)$ .

- To obtain that  $\Phi_1(\alpha) = (S, \Pi)$  we need to find 5 elements  $(P_i)$  of  $GL(V_6)$  such that  ${}^tP_i.M.P_i = \begin{pmatrix} 0 & A_i \\ -A_i & 0 \end{pmatrix}$  where  $A_i$  are 3 by 3 symmetric matrices with linear entries.

We found the following ones easily,

$$P_4 = Id, P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{v}{u} & 0 & 0 & \frac{(-a_{024}-a_{124}+a_{234})}{u} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 + a_{124} - a_{134} & 0 \\ 0 & 0 & \frac{-1}{a_{124}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

but the next ones only after understanding that we should use the  $SL_2 \times SL_2 \times SL_2$  action that preserves the 3 marked lines in the intersection of the two Segre  $\mathbb{P}_1 \times \mathbb{P}_2$  defined by  $w_i = 0$  and  $w_4 = 0$

$$P_1 = \begin{pmatrix} 0 & 0 & \frac{-a_{024}}{a_{234}} & 0 & 0 & 0 \\ \frac{(-u)(a_{024}+u)}{a_{024}(u+1)} & 1 & \frac{a_{024}}{a_{234}} & \frac{a_{024}a_{134}-a_{024}u-a_{024}+ua_{234}}{a_{024}(u+1)} & \frac{-a_{124}}{u} & 0 \\ \frac{u(a_{024}+u)}{a_{024}(u+1)} & 0 & 0 & \frac{-a_{024}a_{134}+a_{024}u+a_{024}-ua_{234}}{a_{024}(u+1)} & 0 & 0 \\ 0 & 0 & \frac{u}{a_{234}} & 0 & 0 & -1 \\ 0 & 0 & \frac{-u}{a_{234}} & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \frac{1}{a_{134}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+v}{u+1} & 0 & 0 & \frac{-v-a_{024}+a_{234}+va_{134}}{u+1} & 0 \\ -\frac{1}{a_{134}} & \frac{-1-v}{u+1} & 1 & 0 & \frac{v+a_{024}-a_{234}-va_{134}}{u+1} & a_{124} \\ \frac{1}{a_{134}} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{a_{134}} & 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

$$P_0 = \begin{pmatrix} a_{124} & \frac{-(a_{024}u+a_{024}-ua_{234})^2}{u^2a_{234}} & -a_{134} & \frac{-a_{124}a_{024}}{a_{024}u+a_{024}-ua_{234}} & 0 & \frac{a_{024}a_{134}-a_{024}u-a_{024}+ua_{234}}{a_{024}u+a_{024}-ua_{234}} \\ 0 & \frac{(a_{024}u+a_{024}-ua_{234})^2}{u^2a_{234}} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{134} & 0 & 0 & \frac{-a_{024}a_{134}+a_{024}u+a_{024}-ua_{234}}{a_{024}u+a_{024}-ua_{234}} \\ 1 & \frac{(u+1)(a_{024}u+a_{024}-ua_{234})}{a_{234}u} & 0 & \frac{u(a_{024}-a_{234})}{a_{024}u+a_{024}-ua_{234}} & -1 & -1 \\ 0 & \frac{(-u-1)(a_{024}u+a_{024}-ua_{234})}{a_{234}u} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and we have proved the theorem 4.6.  $\square$

We are now able to obtain a more explicit version of the theorem 1.3 stated in the introduction.

**Corollary 4.7** *The maps  $\Phi_1$  and  $\Phi_2$  coincide on a open set, and they give birational maps:*

$$\mathbb{P}(W_4 \otimes \bigwedge^2 V_6) / PGL(V_6) \dashrightarrow \mathcal{H}.$$

*Proof:*

First remark that both spaces are irreducible of dimension 24. Now consider with the notations of the definition 4.5 the following map:

$$\begin{aligned} PGL_4 \times \mathcal{A}'_9 &\rightarrow \mathbb{P}(W_4 \otimes \bigwedge^2 V_6) \\ (P, a, b, u) &\mapsto (M_{0123} + w_4 \cdot M_4) \cdot \mathbb{C} \end{aligned}, \text{ where } w_4 = \sum_{i=0}^3 \frac{b_4 \cdot w_i}{b_i}, \text{ and } \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = P \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

and denote by  $f$  its composition with the canonical projection from  $\mathbb{P}(W_4 \otimes \bigwedge^2 V_6)$  to  $\mathbb{P}(W_4 \otimes \bigwedge^2 V_6) / PGL(V_6)$ .

The map  $PGL_4 \times \mathcal{A}'_9 \rightarrow PGL_4 \times \mathcal{A}_9$  has degree 2 because of the permutation of  $u$  and  $v$ , and the rational map  $PGL_4 \times \mathcal{A}_9$  to  $\mathcal{H}$  has degree 5! from the choice of the order and the lemma 4.4. So we have from the theorem 4.6 the commutative diagram of rational maps:

$$\begin{array}{ccccc} PGL_4 \times \mathcal{A}'_9 & \xrightarrow{2:1} & PGL_4 \times \mathcal{A}_9 & \xrightarrow{1:1} & \mathcal{H}_{ord} \\ \downarrow f & & & & \downarrow (5!):1 \\ \mathbb{P}(W_4 \otimes \bigwedge^2 V_6) / PGL(V_6) & & & \xrightarrow[\Phi_2]{\Phi_1} & \mathcal{H} \end{array}$$

So  $\Phi_1$  and  $\Phi_2$  are dominant and coincide on an open set, and we just have to prove that  $f$  has degree  $2 \cdot (5!)$  also. We will do this by providing an example of  $(S, \Pi) \in \mathcal{H}$  such that the permutation of  $u$  and  $v$ , and the permutations of the elements of  $\Pi$  can be obtained by the action of  $GL(V_6)$ . It is more convenient to take an example where all the elements in the preimage of  $(S, \Pi)$  in  $PGL_4 \times \mathcal{A}'_9$  have all the same values for  $(a)$  and  $(b)$ . So we end the proof with the following invariant example:

**Example 4.8** (*Klein-Sylvester*) With the following values:  $u = e^{\frac{2i\pi}{3}}, v = e^{\frac{-2i\pi}{3}}$ . for  $0 \leq i < j < k \leq 4$ ,  $a_{i,j,k} = 1$ . The permutation of  $u$  with  $v$ , and also the permutations of the  $(w_i)_{0 \leq i \leq 4}$  can be obtained from the action of  $GL(V_6)$  on  $M = M_{0123} + w_4.M_4$ . Note that if we add the conditions  $b_i = -b_4$  for  $0 \leq i \leq 3$ , this is the case of the Klein cubic with its Sylvester Pentahedron).

*Proof:* Denote by  $P_T = I_3 \otimes \begin{pmatrix} t_0 & t_1 \\ t_2 & t_3 \end{pmatrix}$  the matrix  $\begin{pmatrix} t_0 & 0 & 0 & t_1 & 0 & 0 \\ 0 & t_0 & 0 & 0 & t_1 & 0 \\ 0 & 0 & t_0 & 0 & 0 & t_1 \\ t_2 & 0 & 0 & t_3 & 0 & 0 \\ 0 & t_2 & 0 & 0 & t_3 & 0 \\ 0 & 0 & t_2 & 0 & 0 & t_3 \end{pmatrix}$  and remark

that  ${}^tP_T M_{0123} P_T = M_{0123}$  when  $\begin{vmatrix} t_0 & t_1 \\ t_2 & t_3 \end{vmatrix} = 1$ . For a square matrix  $\mathcal{T}$ , let  $D_{\mathcal{T}}$  be the block diagonal matrix  $\begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T} \end{pmatrix}$ . So we will first use matrices like  $D_{\mathcal{T}}$  to obtain the desired form in the plane  $w_4 = 0$  and then correct the last matrix with  $P_T$ . We found the following matrices:

. Permutation of  $u$  and  $v$ :  $P_{uv} = I_3 \otimes \begin{pmatrix} \frac{i.u}{\sqrt{2}} & \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{6}}{2} & \frac{i.v}{\sqrt{2}} \end{pmatrix}$  then  ${}^tP_{uv} M_4 P_{uv} = \overline{M_4}$ .

.  $T_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , and  $P_{01} = I_3 \otimes \begin{pmatrix} \frac{u}{\sqrt{2}} & \frac{v+2}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{2}} \end{pmatrix}$ , the conjugaison  ${}^t(D_{T_{01}}.P_{01}).M.(D_{T_{01}}.P_{01})$  permutes  $w_0$  and  $w_1$ .

.  $T_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $P_{02} = I_3 \otimes \begin{pmatrix} \frac{e^{\frac{i\pi}{3}}}{\sqrt{2}} & \frac{-i\sqrt{6}}{2} \\ \frac{e^{\frac{-i\pi}{3}}}{\sqrt{2}} & \frac{v}{\sqrt{2}} \end{pmatrix}$ , the conjugaison  ${}^t(D_{T_{02}}.P_{02}).M.(D_{T_{02}}.P_{02})$  permutes  $w_0$  and  $w_2$ .

.  $T_{03} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ ,  $P_{03} = I_3 \otimes \begin{pmatrix} \frac{i\sqrt{6}}{2} & \frac{-u}{\sqrt{2}} \\ -\frac{v}{\sqrt{2}} & \frac{-i\sqrt{6}}{2} \end{pmatrix}$ , the conjugaison  ${}^t(D_{T_{03}}.P_{03}).M.(D_{T_{03}}.P_{03})$  permutes  $w_0$  and  $w_3$ .

.  $T_{34} = \begin{pmatrix} 0 & -\frac{1}{v} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $P_{34} = I_3 \otimes \begin{pmatrix} -\frac{v}{\sqrt{2}} & -\frac{i\sqrt{6}}{2} \\ -\frac{i\sqrt{6}}{2} & \frac{u}{\sqrt{2}} \end{pmatrix}$ , then  ${}^t(P_3 D_{T_{34}} P_{34}).M.(P_3 D_{T_{34}} P_{34})$  permutes  $w_4$  and  $w_3$  with the matrix  $P_3$  defined in the theorem 4.6.

This completes the proof because we have provided a generating set of the permutations.  $\square$

So the corollary 4.7 and the theorem 1.3 are also proved.  $\square$

## A normal form for 5 general lines in $\mathbb{P}_5$

The explicit forms of the definition 4.5 and the theorem 4.6 have the the following straightforward translation, that should help to handle 5 lines in  $\mathbb{P}_5$  or to understand  $(G(2, V_6))^5/PGL(V_6)$ .

**Corollary 4.9** *Five lines in general position in  $\mathbb{P}_5$  can be put in the following form:*

$$\begin{aligned} & \epsilon_0 \wedge \epsilon_3, \quad \epsilon_1 \wedge \epsilon_4, \quad \epsilon_2 \wedge \epsilon_5, \quad (\epsilon_0 + \epsilon_1 + \epsilon_2) \wedge (\epsilon_3 + \epsilon_4 + \epsilon_5) \\ & (-\epsilon_0 + v\epsilon_4 - \epsilon_5) \wedge (u\epsilon_1 - \epsilon_2 + a_{1,2,4}\epsilon_3 + e_1\epsilon_4 + e_2\epsilon_5) \end{aligned}$$

for some basis  $(\epsilon_i)_{0 \leq i \leq 5}$  of  $V_6$ , and some complex parameters  $u, v, a_{1,2,4}, e_1, e_2$ .

*Proof:* Let's use again the notations of the proposition 4.1. From five general lines in  $\mathbb{P}_5$ , we obtain a five dimensional subspace  $W_5^\vee$  of  $\bigwedge^2 V_6$  containing the corresponding decomposable elements. So choose a general four dimensional vector subspace  $W_4^\vee$  of  $W_5^\vee$ , then  $(W_4^\vee, W_5^\vee)$  is a general element of the incidence variety  $I_{4,5}$ . So from the Theorem 4.6 and the Corollary 4.7 the corresponding element of  $W_5 \otimes \bigwedge^2 V_6$  can be written with the notation of the definition 4.5:  $M_{0123} + w_4.M_4$ . So we obtain the proposition.  $\square$

## 4.3 Questions on the magic square

**Remark 4.10** *Let  $X$  be a non degenerate subvariety of  $\mathbb{P}_{n-1}$ . Then the projection of  $X$  from a general linear space of dimension  $d-2$  is expected to have a finite number  $n_{d,X}$  of points of multiplicity  $d$  when:*

$$d^2 + d(\dim(X) - n - 1) + n = 0.$$

The varieties related to the magic square are famous solutions of this problem for  $d = 2$  or  $d = 3$  with  $n_{d,X} = 1$ . For those varieties, what is the number  $n_{\frac{n}{d},X}$ ?

For the Veronese surface we have  $n_{2,X} \neq n_{3,X}$ , but for  $\mathbb{P}_2 \times \mathbb{P}_2, v_3(\mathbb{P}_1), \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$  (Cf [H]) we have  $n_{d,X} = n_{\frac{n}{d},X} = 1$ . And now, from the proposition 4.1 this equality is also true for  $G(2, 6)$ .

## 5 Applications

Let  $V_{10}$  be a 10-dimensional vector space over the complex numbers. In this section, we will first explain the relationship between two known constructions associated to the choice of a general element of  $\bigwedge^3 V_{10}$ . Then we will discuss how the results of the previous section should be related to the symplectic form of the varieties constructed in [D-V].



## 5.1 Peskine's example in $\mathbb{P}_9$

This example was constructed by C. Peskine to obtain a smooth non quadratically normal variety of codimension 3.

Let  $\mathbb{P}_9$  be a 9 dimensional projective space over the complex numbers, and denote by  $V_{10}$  the vector space  $V_{10} = H^0(\mathcal{O}_{\mathbb{P}_9}(1))$ . Let  $\alpha$  be a general element of  $\bigwedge^3 V_{10}$ , and denote by  $\Omega_{\mathbb{P}_9}^i$  the  $i$ -th exterior power of the cotangent sheaf of  $\mathbb{P}_9$ . From the identification  $\bigwedge^3 V_{10} = H^0(\Omega_{\mathbb{P}_9}^2(3))$ , we obtain a skew-symmetric map  $M_\alpha$  from  $(\Omega_{\mathbb{P}_9}^1)^\vee(-1)$  to  $\Omega_{\mathbb{P}_9}^1(2)$  and an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_9}(-3) \longrightarrow (\Omega_{\mathbb{P}_9}^1)^\vee(-1) \xrightarrow{M_\alpha} \Omega_{\mathbb{P}_9}^1(2) \longrightarrow I_{Y_\alpha}(4) \longrightarrow 0$$

where  $I_{Y_\alpha}$  is the ideal of the smooth variety of dimension 6 defined by the 8 by 8 pfaffians of  $M_\alpha$ . The following proposition is directly deduced from the previous exact sequence.

**Proposition 5.1** *The variety  $Y_\alpha$  is such that  $h^1(I_{Y_\alpha}(2)) = 1$  and its canonical sheaf is  $\omega_{Y_\alpha} = \mathcal{O}_{Y_\alpha}(-3)$ .*

## 5.2 Debarre-Voisin's manifold as a parameter space

Denote by  $G(6, V_{10}^\vee)$  the Grassmannian of 6 dimensional subspaces of  $V_{10}^\vee$ . Let  $K_6$  (resp.  $Q_4$ ) be the tautological sub-bundle (resp. quotient bundle). For any  $p \in G(6, V_{10}^\vee)$ , the corresponding 5-dimensional projective subspace of  $\mathbb{P}_9$  will be denoted by  $\kappa_p$ .

Debarre and Voisin proved in [D-V] the following:

**Theorem 5.2** ([D-V] Th 1.1). *Let  $\alpha$  be a general element of  $\bigwedge^3 V_{10} = H^0(\bigwedge^3 K_6^\vee)$ . The subvariety  $Z_\alpha$  of  $G(6, V_{10}^\vee)$  defined by the vanishing locus of the section  $\alpha$  of  $\bigwedge^3 K_6^\vee$  is an irreducible hyper-Kähler manifold of dimension 4 and second betti number 23.*

We can now remark the following relation between  $Y_\alpha$ ,  $Z_\alpha$  and Palatini threefolds:

**Proposition 5.3** *Let  $p$  be a general element of  $Z_\alpha$ . The scheme defined by the intersection  $Y_\alpha \cap \kappa_p$  is a Palatini threefold.*

*Proof:* The restriction of  $\Omega_{\mathbb{P}_9}^1(1)$  to  $\kappa_p$  is  $\Omega_{\kappa_p}^1(1) \oplus 4\mathcal{O}_{\kappa_p}$ . The vanishing of the restriction of  $\alpha$  to  $\kappa_p$  implies that the restriction of  $M_\alpha$  to  $\kappa_p$  is:  $\begin{pmatrix} 0 & \alpha_p \\ -t_{\alpha_p} & \beta \end{pmatrix}$  with respect to the direct sums:  $(\Omega_{\kappa_p}^1)^\vee(-1) \oplus 4\mathcal{O}_{\kappa_p} \rightarrow (\Omega_{\kappa_p}^1)(2) \oplus 4\mathcal{O}_{\kappa_p}(1)$ . So the ideal generated by the pfaffians of size 8 of this map is also the ideal generated by the maximal minors of  $\alpha_p : 4\mathcal{O}_{\kappa_p} \rightarrow (\Omega_{\kappa_p}^1)(2)$ . In conclusion the scheme defined by the intersection  $Y_\alpha \cap \kappa_p$  is a Palatini threefold as in the remark 3.3.  $\square$

Moreover, the following construction globalize the definition 3.1 and the pfaffian cubic surface over  $Z_\alpha$ .

**Remark 5.4** *The restriction of the bundle  $\bigwedge^2 K_6^\vee \otimes Q_4^\vee$  to  $Z_\alpha$  has a non trivial section. It gives an injective map:*

$$(Q_4)_{|Z_\alpha} \longrightarrow (\bigwedge^2 K_6^\vee)_{|Z_\alpha}$$

*Proof:* The section  $\alpha$  of  $(\bigwedge^3 K_6^\vee)$  gives a map from  $K_6$  to  $(\bigwedge^2 K_6^\vee)$ . But the restriction of this map to  $Z_\alpha$  is zero, so it induces a map from the quotient  $(Q_4)_{|Z_\alpha}$  to  $(\bigwedge^2 K_6^\vee)_{|Z_\alpha}$ . The injectivity of this maps of  $\mathcal{O}_{Z_\alpha}$ -modules follows from the assumption that  $\alpha$  is general.  $\square$

### 5.3 Conjectures on the symplectic form on $Z_\alpha$

**Remark 5.5** *Let  $p$  be a general element of  $Z_\alpha$ . The tangent space  $\mathcal{T}_{(Z_\alpha, p)}$  to  $Z_\alpha$  at  $p$  contains a canonical set of 5 vector spaces of dimension 2.*

*Proof:* Let  $p$  be a general point of  $Z_\alpha$ . From the remark 5.4, the fiber  $Q_{4,p}$  is a 4-dimensional subspace of  $(\bigwedge^2 K_{6,p}^\vee)$ . From the proposition 4.1, we obtain in  $K_{6,p}^\vee$ , a canonical set of five vector subspaces  $(L_i)_{0 \leq i \leq 4}$  of dimension 2 such that  $\bigoplus_{0 \leq i \leq 4} \bigwedge^2 L_i$  contains  $Q_{4,p}$ . So the restriction of the map:

$$m_{21} : \bigwedge^2 K_{6,p}^\vee \otimes K_{6,p}^\vee \rightarrow \bigwedge^3 K_{6,p}^\vee \tag{7}$$

gives the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{T}_{(Z_\alpha, p)} & \longrightarrow & Q_{4,p} \otimes K_{6,p}^\vee & \longrightarrow & \bigwedge^3 K_{6,p}^\vee \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & (\bigoplus_{0 \leq i \leq 4} \bigwedge^2 L_i) \otimes K_{6,p}^\vee & \longrightarrow & \bigwedge^3 K_{6,p}^\vee \longrightarrow 0 \end{array}$$

where the vertical maps are injectives and the first row is the normal sequence of  $Z_\alpha$  in  $G(6, V_{10}^\vee)$  at the point  $p$ . Now remark that  $m_{21}$  vanishes on each  $\bigwedge^2 L_i \otimes L_i$  because  $L_i$  has dimension 2. So we can identify the kernel of the second row of the previous diagram with the 10-dimensional vector space  $\bigoplus_{0 \leq i \leq 4} \bigwedge^2 L_i \otimes L_i$ , and we obtain an injection:

$$\mathcal{T}_{(Z_\alpha, p)} \hookrightarrow \bigoplus_{0 \leq i \leq 4} \bigwedge^2 L_i \otimes L_i.$$

So in general, the kernel of each projection  $\mathcal{T}_{(Z_\alpha, p)} \rightarrow \bigwedge^2 L_i \otimes L_i$  gives a 2 dimensional vector subspace of  $\mathcal{T}_{(Z_\alpha, p)}$ .  $\square$

Now we can remark that five points of  $G(2, \mathcal{T}_{Z_\alpha, p})$  should define an hyperplane  $\gamma$  in  $\bigwedge^2 \mathcal{T}_{Z_\alpha, p}$ . Some random examples with [Macaulay2] let us expect that the ideal of these five lines  $(l_i)$  in  $\mathbb{P}(\mathcal{T}_{Z_\alpha, p})$  is given by the maximal minors of the map:

$$K_{6,p} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{T}_{Z_\alpha, p})} \rightarrow Q_{4,p} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{T}_{Z_\alpha, p})}(1)$$

obtained from the inclusion of the tangent space to  $Z_\alpha$  in the tangent space to  $G(6, V_{10}^\vee)$ . But if the alternate form  $\gamma$  was degenerated, its kernel would give a line in  $\mathbb{P}(\mathcal{T}_{Z_\alpha, p})$  intersecting each  $l_i$ . But a variety defined by quartic hypersurfaces can't have a 5-secant line, so we can expect the following:

**Conjecture 5.6** *The five vector spaces of dimension 2 canonically defined in the remark 5.5 are maximal isotropic subspaces for the symplectic form on  $\mathcal{T}_{Z_\alpha}$  constructed by Debarre and Voisin.*

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